Visual Servoing of Quadrotors for Perching by Hanging from Cylindrical Objects

Supplementary Material, 12/4/2015

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I. Introduction

This work provides additional derivations and technical proofs related to [1] and is considered supplementary material.

II. Preliminaries

The translational dynamics of a quadrotor can be expressed as

$$m\ddot{\mathbf{x}}_q = -mg\mathbf{e}_3 + fR_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 \tag{1}$$

where m is the mass of the vehicle, g is gravity, and f is the net thrust. Further, the angular dynamics are given by

$$\dot{R}_{\mathcal{B}}^{\mathcal{W}} = R_{\mathcal{B}}^{\mathcal{W}}\hat{\mathbf{\Omega}} \tag{2}$$

$$\mathcal{I}\dot{\mathbf{\Omega}} + \mathbf{\Omega}\times\mathcal{I}\mathbf{\Omega} = \mathbf{M} \tag{3}$$

where $\mathbf{\Omega}\in\mathbb{R}^3$ is the angular velocity of the robot in the body frame, $\mathcal{I}$ is the inertial tensor, $M\in\mathbb{R}^3$ is the control moments, and $\hat{\cdot}:\mathbb{R}^3\mapsto\mathfrak{so}(3)$ is defined such that $\hat{\mathbf{a}}\mathbf{b} = \mathbf{a}\times\mathbf{b}$.

Let

$$\mathbf{f}_{des} = m\left(g\mathbf{e}_3 + J^{-1}\left(k_x\mathbf{e}_s + k_v\dot{\mathbf{e}}_s + \ddot{\mathbf{s}}_{des}\right) + \dot{\overline{J^{-1}}}\dot{\mathbf{s}}\right) \tag{4}$$

where

$$\mathbf{e}_s = \mathbf{s}_{des} - \mathbf{s},\ \dot{\mathbf{e}}_s = \dot{\mathbf{s}}_{des} - \dot{\mathbf{s}}$$

are the position and velocity errors in the image coordinates, and k_x and k_v are positive gains. Then, let the thrust be

$$f = \mathbf{f}_{des}\cdot R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 \tag{5}$$

and the attitude controller be as defined in [2], the system defined by eq. (1) and eq. (3) is exponentially stable.

III. Stability of the Angular Dynamics

Following the treatment in [3], the Lyapunov candidate is

$$\mathcal{V}_R = \frac{1}{2}\mathbf{e}_\Omega\cdot\mathcal{I}\mathbf{e}_\Omega + k_R\Psi\left(R, R_d\right) + c_2\mathbf{e}_R\cdot\mathbf{e}_\Omega, \tag{6}$$

with c_2 being a positive scalar, such that,

$$\mathbf{z}_\theta^T M_\theta\mathbf{z}_\theta \le \mathcal{V}_R \le \mathbf{z}_\theta^T M_\Theta\mathbf{z}_\theta, \tag{7}$$

$$\dot{\mathcal{V}}_R \le -\mathbf{z}_\theta^T W_\theta\mathbf{z}_\theta, \tag{8}$$

where $\mathbf{z}_\theta = \left[\|\mathbf{e}_R\|, \|\mathbf{e}_\Omega\|\right]^T$, and M_θ, M_Θ, and W_θ are positive definite. This in turn guarantees the asymptotic staibility of the attitude dynamics.

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IV. Stability of Translational Dynamics

The velocity of the image features is given by

$$\dot{\mathbf{s}} = \frac{\partial\Gamma\left(\mathbf{P}_1^{\mathcal{V}}\right)}{\partial\mathbf{P}_1^{\mathcal{V}}}\dot{\mathbf{P}}_1^{\mathcal{V}}.$$

Since

$$\dot{\mathbf{P}}_1^{\mathcal{V}} = -R_{\mathcal{C}}^{\mathcal{V}}R_{\mathcal{W}}^{\mathcal{C}}\dot{\mathbf{x}}_q,$$

we can express the image feature velocities in terms of the robot velocity in the inertial frame and the point $\mathbf{P}_1^{\mathcal{V}}$

$$\dot{\mathbf{s}} = -\frac{\partial\Gamma\left(\mathbf{P}_1^{\mathcal{V}}\right)}{\partial\mathbf{P}_1^{\mathcal{V}}}R_{\mathcal{C}}^{\mathcal{V}}R_{\mathcal{W}}^{\mathcal{C}}\dot{\mathbf{x}}_q \equiv J\dot{\mathbf{x}}_q. \tag{9}$$

Using (9), the acceleration of the robot can be expressed in terms of J, $\mathbf{s}$, and their derivatives

$$m\left(J^{-1}\ddot{\mathbf{s}} + \dot{\overline{J^{-1}}}\dot{\mathbf{s}}\right) = fR_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 - mg\mathbf{e}_3. \tag{10}$$

Rearranging, the dynamics of the image features are

$$\ddot{\mathbf{s}} = \frac{1}{m}J\left(fR_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 - mg\mathbf{e}_3 - m\dot{\overline{J^{-1}}}\dot{\mathbf{s}}\right). \tag{11}$$

Using (9), we can determine the image errors

$$\ddot{\mathbf{e}}_s = \ddot{s}_{des} - \frac{1}{m}J\left(fR_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 - mg\mathbf{e}_3 - m\dot{\overline{J^{-1}}}\dot{\mathbf{s}}\right), \tag{12}$$

so that

$$m\ddot{\mathbf{e}}_s = m\ddot{\mathbf{s}}_{des} - fJR_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3 + Jmg\mathbf{e}_3 + mJ\dot{\overline{J^{-1}}}\dot{\mathbf{s}}. \tag{13}$$

Defining

$$\mathbf{X} = J\frac{f}{\mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3}\left(R_c\mathbf{e}_3 - \left(\mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3\right)R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3\right), \tag{14}$$

the error dynamics become

$$\begin{aligned} m\ddot{\mathbf{e}}_s =& m\ddot{\mathbf{s}}_{des} - J\left(\frac{f}{\mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3}R_c\mathbf{e}_3\right) + \mathbf{X} \\ &+ Jmg\mathbf{e}_3 + mJ\dot{\overline{J^{-1}}}\dot{\mathbf{s}}. \end{aligned} \tag{15}$$

Next, let

$$f = \mathbf{f}_{des}\cdot R_{\mathcal{B}}^{\mathcal{W}}\mathbf{e}_3, \tag{16}$$

and the commanded attitude be defined by

$$R_c\mathbf{e}_3 = \frac{\mathbf{f}_{des}}{\|\mathbf{f}_{des}\|}. \tag{17}$$

Then, from the previous two equations, we have

$$f = \|\mathbf{f}_{des}\| \, \mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}} \mathbf{e}_3. \tag{18}$$

Substituting this into (15) and using $\mathbf{f}_{des}$, we have

$$\begin{aligned} m\ddot{\mathbf{e}}_s =& m\ddot{\mathbf{s}}_{des} - J\left(\frac{\|\mathbf{f}_{des}\| \, \mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}} \mathbf{e}_3}{\mathbf{e}_3^T R_c^T R_{\mathcal{B}}^{\mathcal{W}} \mathbf{e}_3} R_c \mathbf{e}_3\right) + \mathbf{X} \\ &+ Jmg\mathbf{e}_3 + mJ\dot{J^{-1}}\dot{\mathbf{s}} \\ =& m\ddot{\mathbf{s}}_{des} - J\left(\|\mathbf{f}_{des}\| \, R_c\mathbf{e}_3\right) + \mathbf{X} \\ &+ Jmg\mathbf{e}_3 + mJ\dot{J^{-1}}\dot{\mathbf{s}} \\ =& m\ddot{\mathbf{s}}_{des} - J\mathbf{f}_{des} + \mathbf{X} + Jmg\mathbf{e}_3 + mJ\dot{J^{-1}}\dot{\mathbf{s}}, \end{aligned} \tag{19}$$

From eq. (4) the error equation finally becomes

$$m\ddot{\mathbf{e}}_s = -k_x\mathbf{e}_s - k_v\dot{\mathbf{e}}_s + \mathbf{X}. \tag{20}$$

Proof: We use the Lyapunov candidate from [3]

$$\mathcal{V}_s = \frac{1}{2}k_x\|\mathbf{e}_s\|^2 + \frac{1}{2}m\|\dot{\mathbf{e}}_s\|^2 + c_1\mathbf{e}_s\cdot\dot{\mathbf{e}}_s. \tag{21}$$

Now, let $\mathbf{z}_s = [\ \|\mathbf{e}_s\|, \|\dot{\mathbf{e}}_s\|\ ]^T$, then it follows that the Lyapunov function $\mathcal{V}_v$ is bounded as

$$\mathbf{z}_s^T M_s \mathbf{z}_s \le \mathcal{V}_v \le \mathbf{z}_s^T M_S \mathbf{z}_s, \tag{22}$$

where $M_s, M_S \in \mathbb{R}^{2\times 2}$ are defined as,

$$M_s = \frac{1}{2}\begin{bmatrix} K_p & -c_1 \\ -c_1 & m \end{bmatrix},\ M_S = \frac{1}{2}\begin{bmatrix} K_p & c_1 \\ c_1 & m \end{bmatrix}. \tag{23}$$

Then,

$$\dot{\mathcal{V}}_s = k_x(\dot{\mathbf{e}}_s\cdot\mathbf{e}_s) + m(\ddot{\mathbf{e}}_s\cdot\dot{\mathbf{e}}_s) + c_1(\mathbf{e}_s\cdot\ddot{\mathbf{e}}_s + \dot{\mathbf{e}}_s\cdot\dot{\mathbf{e}}_s), \tag{24}$$

and incorporating (20),

$$\begin{aligned} \dot{\mathcal{V}}_s =& \frac{c_1k_x}{m}\|\mathbf{e}_s\|^2 + (k_v - c_1)\|\dot{\mathbf{e}}_s\|^2 \\ &+ c_1\tfrac{K_v}{m}(\mathbf{e}_s\cdot\dot{\mathbf{e}}_s) + \mathbf{X}\cdot\left(\tfrac{c_1}{m}\mathbf{e}_s + \dot{\mathbf{e}}_s\right). \end{aligned} \tag{25}$$

Now, we establish a bound on $\mathbf{X}$. From (14),

$$\begin{aligned} \mathbf{X} =& J\frac{f}{\mathbf{e}_3^T R_c^T R\mathbf{e}_3}\left(\left(\mathbf{e}_2^T R_c^T R\mathbf{e}_3\right)R\mathbf{e}_3 - R_c\mathbf{e}_3\right) \\ \|\mathbf{X}\| \le& \|J\| \left\| \frac{\|\mathbf{f}_{des}\| \, R_c\mathbf{e}_3\cdot R\mathbf{e}_3}{R_c\mathbf{e}_3\cdot R\mathbf{e}_3} \right\| \|e_R\| \\ \le& \|J\| \, \|\mathbf{f}_{des}\| \, \|e_R\| \\ \le& \|J\| \, m \left\| g e_3 + J\left(k_x\mathbf{e}_s + k_v\dot{\mathbf{e}}_s + \ddot{\mathbf{s}}_{des}\right) + \dot{J^{-1}}\dot{\mathbf{s}} \right\| \|e_R\| \\ \le& \left(k_x'\|\mathbf{e}_s\| + k_v'\|\dot{\mathbf{e}}_s\| + B\right)\|e_R\|, \end{aligned} \tag{26}$$

where k_x', k_v', B are as defined as

$$k_x' = m\|J\|\,\|J\|\,k_x, \tag{27}$$

$$k_v' = m\left(\|J\|\,\|J\|\,k_v + \left\|\dot{J^{-1}}\right\|\right), \tag{28}$$

$$B = m\|J\|\left(g + \|J\|\,\|\ddot{\mathbf{s}}_{des}\| + \left\|\dot{J^{-1}}\right\|\,\|\dot{\mathbf{s}}_{des}\|\right), \tag{29}$$

and from [3], $0 \le \|e_R\| \le 1$.

Next we will show that there exists positive constants $\gamma_1, \gamma_2, \gamma_3$ s.t., $\|J\| \le \gamma_1$, $\|J^{-1}\| \le \gamma_2$, and $\left\|\dot{J^{-1}}\right\| \le \gamma_3$. Since Γ is smooth (we only require C^2 here), J is smooth on the closed set S. This implies J is bounded on S, i.e., $\exists\gamma_1 > 0$, s.t. $\|J\| < \gamma_1$. Next, since J is smooth and nonsingular on S, the inverse is well defined and is smooth on S, which implies J^{-1} is bounded on S, i.e., $\exists\gamma_2 > 0$, s.t. $\|J^{-1}\| < \gamma_2$. Next, observe that $\frac{d}{dt}J^{-1}(\mathbf{x}_q) = \frac{\partial}{\partial\mathbf{x}_q}J^{-1}(\mathbf{x}_q)\dot{\mathbf{x}}_q$ is a composition of smooth functions on S, implying that it is bounded on S, i.e., $\exists\gamma_3 > 0$, s.t. $\left\|\dot{J^{-1}}\right\| < \gamma_3$.

Then, similar to [4], we can express $\dot{\mathcal{V}}_v$ as

$$\begin{aligned} \dot{\mathcal{V}}_s =& \frac{c_1k_x}{m}\|\mathbf{e}_s\|^2 + (k_v - c_1)\|\dot{\mathbf{e}}_s\|^2 \\ &+ c_1\tfrac{K_v}{m}(\mathbf{e}_s\cdot\dot{\mathbf{e}}_s) + \mathbf{X}\cdot\left(\tfrac{c_1}{m}\mathbf{e}_s + \dot{\mathbf{e}}_s\right) \\ \le& -\left[\ \|\mathbf{e}_s\| \quad \|\dot{\mathbf{e}}_s\|\ \right] W_{s_1} \begin{bmatrix} \|\mathbf{e}_s\| \\ \|\dot{\mathbf{e}}_s\| \end{bmatrix} \\ &+ k_p'\|\mathbf{e}_s\|\,\|e_R\|\left(\frac{c_1}{m}\|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\|\right) \\ &+ k_v'\|\dot{\mathbf{e}}_s\|\,\|e_R\|\left(\frac{c_1}{m}\|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\|\right) \\ &+ B\|e_R\|\left(\frac{c_1}{m}\|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\|\right). \end{aligned}$$

This expression can be written as,

$$\dot{\mathcal{V}}_s \le -\mathbf{z}_s^T W_s \mathbf{z}_s + \mathbf{z}_s^T W_{s\theta}\mathbf{z}_\theta, \tag{30}$$

where W_s is defined in eq. (33).

$$W_{s_1} = \begin{bmatrix} \frac{c_1k_x}{m} & \frac{c_1k_v}{2m} \\ \frac{c_1k_v}{2m_q} & k_v - c_1 \end{bmatrix}, W_{s\theta} = \begin{bmatrix} \frac{c_1}{m}B & 0 \\ B & 0 \end{bmatrix}, \tag{31}$$

$$W_{s_2} = \begin{bmatrix} \frac{c_1\alpha k_x'}{m} & \frac{\alpha}{2}\left(\frac{c_1}{m}k_v' + k_x'\right) \\ \frac{\alpha}{2}\left(\frac{c_1}{m}k_v' + k_x'\right) & \alpha k_v' \end{bmatrix}, \tag{32}$$

$$W_s = W_{s_1} - W_{s_2}. \tag{33}$$

Since $W_s = (W_s)^T$ and $W_s \in \mathbb{R}^{2\times 2}$, it is sufficient to show that $\det(W_s) > 0$ and $W_s(1,1) > 0$ in order to claim that $W_s > 0$. Then, assuming $(1 - \alpha\gamma_1\gamma_2 m) > 0$, we have $w_{11} > 0$. This is reasonable since α is a functional on the attitude error such that $\alpha \in [0,1]$. Thus, this assumption is simply a bound on the attitude error. The determinant can be expressed as a quadratic function of k_v such that

$$\det(W_s) = \beta_0 + \beta_1 k_v + \beta_2 k_v^2, \tag{34}$$

and β_i is a function of c_1, k_x, γ_1, γ_2, γ_3, and m. The critical point of the quadratic occurs when

$$k_v = \frac{k_x m}{c_1} + \frac{k_x m + \alpha c_1\gamma_1\gamma_3}{c_1\left(1 - \alpha\gamma_1\gamma_2 m\right)}, \tag{35}$$

and has a value of

$$\det(W_s) = \frac{k_x\left(1 - \alpha\gamma_1\gamma_2 m\right)\left(k_x m - c_1^2\right)}{m}. \tag{36}$$

In both equations, since $(1 - \alpha\gamma_1\gamma_2 m) > 0$, the eq. (35) is positive, eq. (36) is positive and $W_s > 0$.

V. Stability of the Underactuated System

Now, we consider the combined Lyapunov candidate for the translational and rotational error dynamics, $\mathcal{V} = \mathcal{V}_s + \mathcal{V}_R$. From (7) and (22), we have,

$$\mathbf{z}_s^T M_s \mathbf{z}_s + \mathbf{z}_\theta^T M_\theta \mathbf{z}_\theta \leq \mathcal{V} \leq \mathbf{z}_\theta^T M_\Theta \mathbf{z}_\theta + \mathbf{z}_s^T M_S \mathbf{z}_s. \tag{37}$$

Further, we see that

$$\dot{\mathcal{V}} \leq -\mathbf{z}_s^T W_s \mathbf{z}_s + \mathbf{z}_s^T W_{s\theta} \mathbf{z}_\theta - \mathbf{z}_\theta^T W_\theta \mathbf{z}_\theta, \tag{38}$$

$$\leq -\lambda_{min}(W_s) \left\|\mathbf{z}_s\right\|^2 + \|W_{v\theta}\| \, \|\mathbf{z}_s\| \, \|\mathbf{z}_\theta\|$$

$$\leq -\lambda_{min}(W_\theta) \left\|\mathbf{z}_\theta\right\|^2. \tag{39}$$

Suppose that we choose positive constants c_1, k_x, k_v, k_R such that

$$k_x > \frac{c_1^2}{m}, \tag{40}$$

$$\lambda_{min}\left(W_\theta\right) > \frac{4\|W_{s\theta}\|^2}{\lambda_{min}\left(W_s\right)} \tag{41}$$

we have $\dot{\mathcal{V}}$ to be negative definite ensuring the equilibrium of the closed-loop system to be asymptotically stable. ∎

References

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